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On anti-magic labeling for graph products

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Abstract

An anti-magic labeling of a finite simple undirected graph with p vertices and q edges is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all edges incident to such vertex. A graph is called anti-magic if it admits an anti-magic labeling. Hartsfield and Ringel conjectured in 1990 that all connected graphs except K_2 are anti-magic. Recently, Alon et al. showed that this conjecture is true for dense graphs, i.e. it is true for p-vertex graphs with minimum degree $\Omega(\log p)$. In this article, new classes of sparse anti-magic graphs are constructed through Cartesian products and lexicographic products.

Keywords: Anti-magic labeling; Cartesian product; Lexicographic product

1. Introduction

All graphs in this paper are finite, simple, undirected, and without loops unless otherwise stated. In 1990, Hartsfield and Ringel [5] introduced the concepts called anti-magic labeling and anti-magic graphs.

Definition 1. For a graph $G = (V, E)$ with p vertices and q edges and without any isolated vertex, an *anti-magic* edge labeling is a bijection $f : E \to \{1, 2, \ldots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{N}$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. A graph is called anti-magic if it admits an anti-magic labeling.

Hartsfield and Ringel showed that paths, cycles, complete graphs K_n $(n \geqslant 3)$ are anti-magic. They conjectured that all connected graphs besides K_2 are anti-magic, which remains unsettled. Recently, Alon et al. [1] showed that the last conjecture is true for dense graphs. They showed that all graphs with $n(\geqslant 4)$ vertices and minimum degree $\Omega(\log n)$ are anti-magic. They also proved that if G is a graph with $n(\geqslant 4)$ vertices and the maximum degree $\Delta(G) \geqslant 4n - 2$, then G is anti-magic and all complete partite graphs except K_2 are anti-magic. More recently, Hefetz [6] proved that, among others, for $k \in \mathbb{N}$, a graph G with 3^k vertices is anti-magic if it admits a K_3-factor. Also Wang [8] showed that the Cartesian products of cycles and regular graphs are anti-magic, and in particular, higher dimensional torus graphs are anti-magic. In this paper, we consider the anti-magic labeling of Cartesian products and lexicographic products of graphs, and using these constructions we may construct new classes of sparse anti-magic graphs. For more conjectures

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and open problems on anti-magic graphs and related type of graph labeling problems, please see the dynamic survey article of Gallian [4].

In this paper, we introduce new classes of anti-magic graphs through Cartesian and lexicographic products.

2. Preliminaries

It has been proved in [5] that all paths $P_n, n \geqslant 3$, and all cycles $C_n, n \geqslant 3$, are anti-magic. We state here in the following lemmas:

Lemma 2. *The paths P_{n+1} are anti-magic for $n \geqslant 2$.*

We may treat the cycles C_n as the graphs obtained by the paths P_{n+1} through identifying the two end points. Hence an anti-magic labeling of cycle C_n follows from an anti-magic labeling of the path P_{n+1}.

Lemma 3. *The cycles C_n are anti-magic for $n \geqslant 3$.*

In order to obtain an anti-magic labeling of the Cartesian product of graphs, we first consider the translation on an existing anti-magic labeling:

Definition 4. A graph $G = (V, E)$ with $p = |V|$ vertices and $q = |E|$ edges is called *k-anti-magic*, where k is a non-negative integer, if there exists a bijection $f : E \to \{k+1, k+2, \ldots, k+q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{N}$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. Note that the original anti-magic labeling is 0-anti-magic, in particular.

However, usually the labeling obtained from the translation on an anti-magic labeling could lose its anti-magic-ness. Therefore, we find the following sufficient condition to have a k-anti-magic translation by k on an existing anti-magic labeling:

Lemma 5. *Let G be an anti-magic (p, q)-graph with vertex set $\{v_1, \ldots, v_p\}$. Without loss of generality (via renaming vertices), we may assume that the degree sequence is $\deg(v_1) \leqslant \deg(v_2) \leqslant \cdots \leqslant \deg(v_p)$. Suppose $f : E(G) \to \{1, 2, \ldots, q\}$ is an anti-magic edge labeling of G with the property that $f^+(v_1) < f^+(v_2) < \cdots < f^+(v_p)$, where $f^+ : V(G) \to \mathbb{N}$ is the induced vertex sum. That is, the ordering of the vertex sums is consistent with that of the degree sequence. Let $g : E(G) \to \mathbb{N}$ be the k-translation of the edge labeling f defined by $g(e) = f(e) + k$, where the integer $k \geqslant 0$, for each edge e in G, and g^+ is the induced vertex sum from g. Then $g^+(v_1) < g^+(v_2) < \cdots < g^+(v_p)$, i.e., the edge labeling g is k-anti-magic.*

Proof. $g^+(v_i) = \sum(f(uv_i) + k) = f^+(v_i) + k \cdot \deg(v_i) < f^+(v_{i+1}) + k \cdot \deg(v_{i+1}) = \sum(f(uv_{i+1}) + k) = g^+(v_{i+1})$, for $1 \leqslant i \leqslant p-1$. $\square$

Note that if the considered graphs are regular, then the sufficient condition in the above lemma holds trivially. Therefore,

Corollary 6. *The following are true:*

(1) *Suppose the graph G is regular and anti-magic, then it is k-anti-magic, where $k \geqslant 0$.*
(2) *Suppose G_i is regular and anti-magic for each $1 \leqslant i \leqslant n$, then the disjoint union $G_1 + G_2 + \cdots + G_n$ is k-anti-magic, where $k \geqslant 0$. In particular, every 2-regular graph is k-anti-magic, and $nG = G + G + \cdots + G$ is k-anti-magic if G is regular and anti-magic.*

Proof. For part (1), suppose G is regular and anti-magic, then by Lemma 5 the translation by k on an anti-magic labeling of G is k-anti-magic, where the integer $k \geqslant 0$. As for part (2), let $\deg(G_i)$ be the constant degree of the regular graph G_i. Without loss of generality, we may assume that $\deg(G_1) \leqslant \deg(G_2) \leqslant \cdots \leqslant \deg(G_n)$. Also note that from part (1), each G_i is k-anti-magic, hence we may put the edge labels in order to obtain an anti-magic labeling of the disjoint union $G_1 + G_2 + \cdots + G_n$. Again by Lemma 5, the disjoint union is k-anti-magic. $\square$

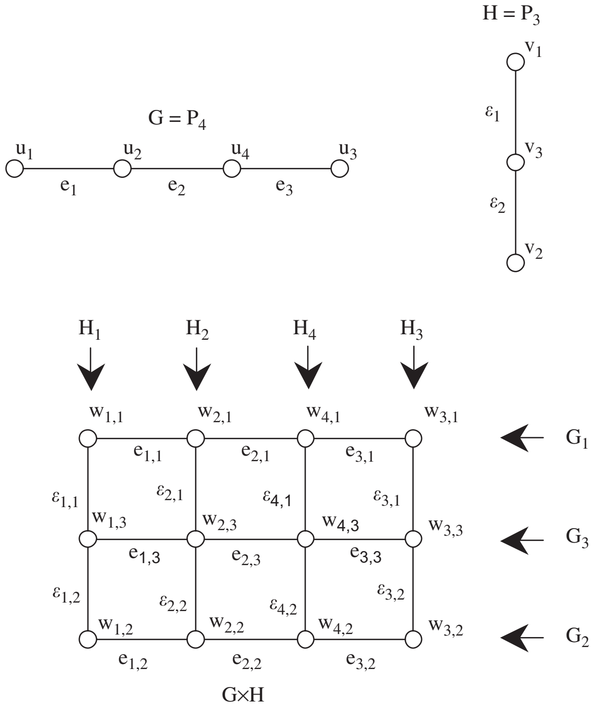

Fig. 1. Conventions of notations for Cartesian product $G \times H$.

For convenience, we will use the following convention of notations throughout later sections for Cartesian products of graphs and associated labeling:

Convention: Given two graphs G and H with $|V(G)| = p_1$, $|E(G)| = q_1$ and $|V(H)| = p_2$, $|E(H)| = q_2$, respectively. The Cartesian product of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, and (u_i, v_j) is adjacent to (u_k, v_l) whenever (1) $u_i = u_k$ and $v_j v_l \in E(H)$, or (2) $v_j = v_l$ and $u_i u_k \in E(G)$. Note that $G \times H$ can be decomposed into p_2 isomorphic slices $G_j's$ of G, together with p_1 isomorphic slices $H_i's$ of H. Namely, the subgraphs $G_j's$ are induced by the vertices (u_i, v_j), where j is fixed and i is running from 1 to p_1, and the subgraphs $H_i's$ are induced by the vertices (u_i, v_j), where i is fixed and j is running from 1 to p_2. In graphs G and H, without loss of generality, for edges $e_i \in E(G)$ and $\varepsilon_j \in E(H)$ there are bijections $g : E(G) \to \{1, 2, \ldots, q_1\}$ given by $g(e_i) = i$, where $1 \leqslant i \leqslant q_1$, and $h : E(H) \to \{1, 2, \ldots, q_2\}$ given by $h(\varepsilon_j) = j$, where $1 \leqslant j \leqslant q_2$, such that the induced vertex sum $g^+ : V(G) \to \mathbb{N}$ has the ordering $g^+(u_1) \leqslant g^+(u_2) \leqslant \cdots \leqslant g^+(u_{p_1})$, and the vertex sum $h^+ : V(H) \to \mathbb{N}$ has the ordering $h^+(v_1) \leqslant h^+(v_2) \leqslant \cdots \leqslant h^+(v_{p_2})$, respectively. In the graph $G \times H$, the vertex $(u_i, v_j) \in V(G) \times V(H)$ is represented by $w_{i,j}$, and the edge e_i in the subgraph G_l is represented by $e_{i,l}$, and the edge ε_j in the subgraph H_k is represented by $\varepsilon_{k,j}$, respectively. Let the bijection $f : E(G \times H) \to \{1, 2, \ldots, p_1 q_2 + p_2 q_1\}$ be an edge labeling of $G \times H$, and the induced vertex sum is f^+. We denote f_G and f_H to be the edge labeling f restricted to the slices of G_l, $1 \leqslant l \leqslant p_2$, and H_k, $1 \leqslant k \leqslant p_1$, respectively. And f_G^+ and f_H^+ represent the induced vertex sums, respectively. Therefore, we have that $f^+(w_{i,j}) = f_G^+(w_{i,j}) + f_H^+(w_{i,j})$, $1 \leqslant i \leqslant p_1$ and $1 \leqslant j \leqslant p_2$.

See Fig. 1 as an example for the conventions mentioned above.

We call the Cartesian product of paths with paths $P_m \times P_n$ a *lattice grid graph*, the Cartesian product of paths and cycles $P_m \times C_n$ a *prism grid graph*, and the Cartesian product of cycles with cycles $C_m \times C_n$ a *toroidal grid graph*.

3. Lattice grids are anti-magic

We would like to show that the lattice grid graphs $P_m \times P_n$ are anti-magic for $m \geqslant n \geqslant 2$. Let us prove some basic facts first.

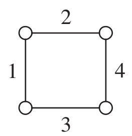

Fig. 2. An anti-magic labeling of $P_2 \times P_2$.

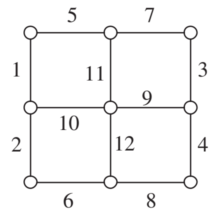

Fig. 3. An anti-magic labeling of $P_3 \times P_3$.

Proposition 7. *The lattice grid graphs $P_m \times P_2$ are anti-magic for $m \geqslant 2$.*

Proof. In this graph $P_m \times P_2$, we denote $G = P_m$ and $H = P_2$. It is easy to see that $P_2 \times P_2$ is anti-magic (see Fig. 2). For the graph $H = P_2$, we have the same vertex sum of two vertices, i.e. $h^+(v_1) = h^+(v_2)$, where h is an edge labeling for H. For $G = P_m$, $m > 2$, we label the edges by $g(e_i) = i$ for $1 \leqslant i \leqslant m-1$ as in Lemma 2, such that $g^+(u_1) < g^+(u_2) < \cdots < g^+(u_m)$. Using the convention of the notations for Cartesian products of graphs, we label the edges of the graph $P_m \times P_2$ for $m > 2$ by

$$f_H(\varepsilon_{k,j}) = k \quad \text{for } 1 \leqslant k \leqslant m, \quad j = 1,$$
$$f_G(e_{i,l}) = m + l + 2 \cdot (i-1) \quad \text{for } 1 \leqslant i \leqslant m-1, \quad l = 1, 2.$$

Then the induced vertex sums can be calculated and have the following ordering:

$f^+(w_{1,1}) < f^+(w_{1,2}) < f^+(w_{2,1}) < f^+(w_{2,2}) < \cdots < f^+(w_{m,1}) < f^+(w_{m,2})$. Therefore an anti-magic labeling is obtained. $\square$

Proposition 8. *The lattice grid graphs $P_m \times P_3$ are anti-magic for $m \geqslant 2$.*

Proof. In this graph $P_m \times P_3$, we denote $G = P_m$ and $H = P_3$. Note that by previous proposition, $P_2 \times P_3$ is anti-magic. Also it is easy to see that $P_3 \times P_3$ is anti-magic (see Fig. 3).

Similarly, we label the edges of the graphs P_m and P_3 by $g(e_i) = i$ for $1 \leqslant i \leqslant m-1$ and $h(\varepsilon_j) = j$ for $j = 1, 2$ as in Lemma 2, such that $g^+(u_1) < g^+(u_2) < \cdots < g^+(u_m)$ and $h^+(v_1) < h^+(v_2) < h^+(v_3)$ respectively. Using the convention of the notations for Cartesian products of graphs, when $m > 3$, we label the edges of the graph $P_m \times P_3$ by

$$f_H(\varepsilon_{k,j}) = \begin{cases} j + 2(k-1), & k = 1, 2, \ j = 1, 2, \\ 3m + 1 + j + 2(k-3), & 3 \leqslant k \leqslant m, \ j = 1, 2, \end{cases}$$

$$f_G(e_{i,l}) = \begin{cases} 4 + l + 2(i-1), & 1 \leqslant i \leqslant m-1, \ l = 1, 2, \\ 2m + 2 + i, & 1 \leqslant i \leqslant m-1, \ l = 3. \end{cases}$$

Then the induced vertex sums are calculated and have the following ordering: $f^+(w_{1,1}) < f^+(w_{1,2}) < f^+(w_{2,1}) < f^+(w_{2,2}) < f^+(w_{1,3}) < f^+(w_{2,3}) < f^+(w_{3,1}) < f^+(w_{3,2}) < f^+(w_{4,1}) < f^+(w_{4,2}) < \cdots < f^+(w_{m,1}) < f^+(w_{m,2}) < f^+(w_{3,3}) < f^+(w_{4,3}) < \cdots < f^+(w_{m,3})$. Therefore an anti-magic labeling is obtained. $\square$

Now we are in a position to prove in general that the lattice grid graph $P_m \times P_n$ is anti-magic, where $m \geqslant n \geqslant 2$.

Theorem 9. *The lattice grid graphs $P_m \times P_n$ are anti-magic for $m \geqslant n \geqslant 2$.*

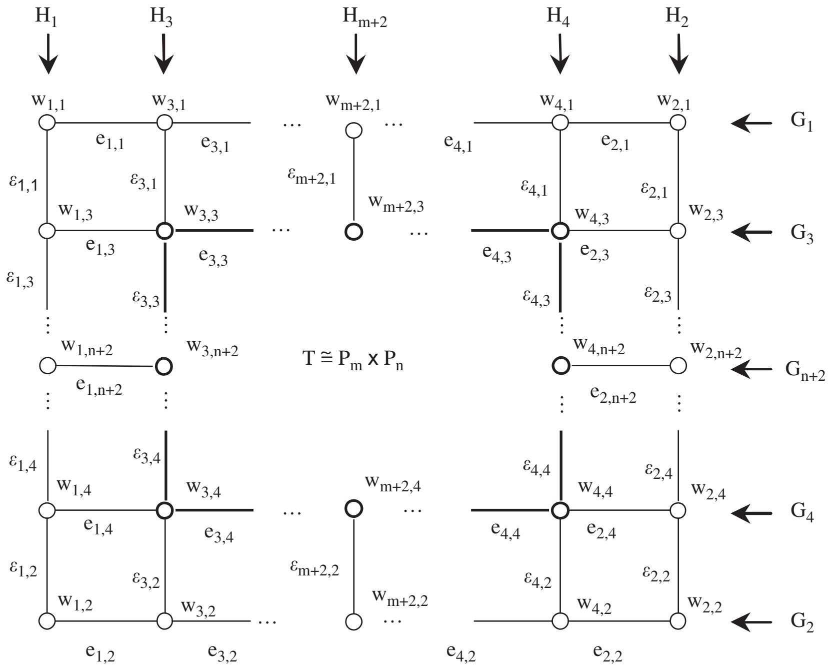

Fig. 4. T $\cong P_m \times P_n$ is an induced subgraph of the graph $P_{m+2} \times P_{n+2}$.

Proof. We prove this theorem by using the induction. The basis step is the facts that $P_m \times P_2$ ($m \geqslant 2$), and $P_m \times P_3$ ($m \geqslant 3$) are anti-magic, which are already shown in previous propositions. Assume $P_m \times P_n$ has an anti-magic labeling with a particular vertex sum ordering satisfying Lemma 5, i.e. one consistent with the degree sequence. Now we proceed the induction on $P_{m+2} \times P_{n+2}$. Assume that the vertex set of the graph P_{m+2} is $\{u_i : 1 \leqslant i \leqslant m+2\}$, and the vertex set of the graph P_{n+2} is $\{v_j : 1 \leqslant j \leqslant n+2\}$. Hence the Cartesian product $P_{m+2} \times P_{n+2}$ is a graph with the vertex set $\{w_{i,j} = (u_i, v_j) : 1 \leqslant i \leqslant m+2, 1 \leqslant j \leqslant n+2\}$ and the edge set $\{e_{i,l} : 1 \leqslant i \leqslant m+1, 1 \leqslant l \leqslant n+2\} \cup \{\varepsilon_{k,j} : 1 \leqslant k \leqslant m+2, 1 \leqslant j \leqslant n+1\}$, using the convention of the notations for Cartesian products of graphs as we state earlier. Let T be the induced subgraph with the vertex set $\{w_{i,j} : i \neq 1, 2, j \neq 1, 2\}$. Note that the graph T is isomorphic to $P_m \times P_n$ as shown in Fig. 4. By the induction hypothesis, $T \cong P_m \times P_n$ is anti-magic, and suppose the anti-magic labeling is $t : E(P_m \times P_n) \to \{1, 2, \ldots, m(n-1) + (m-1)n\}$, and moreover the induced vertex sums $t^+ : V(P_m \times P_n) \to \mathbb{N}$ have the following ordering:

$$
\begin{aligned}
&t^+(w_{3,3}) < t^+(w_{3,4}) < t^+(w_{4,3}) < t^+(w_{4,4}) < t^+(w_{3,5}) < t^+(w_{3,6}) < \cdots < t^+(w_{3,n+2}) < t^+(w_{4,5}) \\
&\quad < t^+(w_{4,6}) < \cdots < t^+(w_{4,n+2}) < t^+(w_{5,3}) < t^+(w_{5,4}) < t^+(w_{6,3}) < t^+(w_{6,4}) < \cdots < t^+(w_{m+2,3}) \\
&\quad < t^+(w_{m+2,4}) < t^+(w_{5,5}) < \cdots < t^+(w_{m+2,n+2}).
\end{aligned}
$$

Note that the ordering of these vertex sums follows exactly the ascending ordering of the degrees of the according vertices (i.e. vertices of degrees 2, 3, 4), just as the situation stated in Lemma 5. Hence by Lemma 5, we know $T \cong P_m \times P_n$ is k-anti-magic. In order to obtain an anti-magic labeling $f : E(P_{m+2} \times P_{n+2}) \to \{1, 2, \ldots, (m+1)(n+2) + (m+2)(n+1)\}$, the plan of attack is first to label the edges in T by an k-anti-magic labeling, where $k = 4(m+n+1)$, i.e. a translation on the anti-magic labeling by k. And secondly to label the remaining edges, i.e. the edges in $E(P_{m+2} \times P_{n+2}) - E(T)$, by integers $1, 2, \ldots, 4(m+n+1) = |E(P_{m+2} \times P_{n+2})| - |E(T)|$. First of all, for the edges of the graph T we shift by $k = 4(m+n+1)$ on the anti-magic edge labeling followed from the induction hypothesis that $T \cong P_m \times P_n$ is anti-magic. That is, for all $e \in E(T)$ we define $f(e) = t(e) + k$, where $k = 4(m+n+1)$. Then, we assign $1, 2, \ldots, 4(m+n+1)$ to the edges in $E(P_{m+2} \times P_{n+2}) - E(T)$ as follows: label the edges of the outer cycles $\varepsilon_{1,1}, \varepsilon_{1,2}, \ldots, \varepsilon_{1,n+1}, \varepsilon_{2,1}, \varepsilon_{2,2}, \ldots, \varepsilon_{2,n+1}, e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, \ldots, e_{m+1,1}, e_{m+1,2}$ using the consecutive integers from 1 to $2(m+n+2)$, then we label the edges connecting the outer cycle and the subgraph T, i.e. $e_{1,3}, e_{1,4}, \ldots, e_{1,n+2}, e_{2,3}, e_{2,4}, \ldots, e_{2,n+2}, \varepsilon_{3,1}, \varepsilon_{3,2}, \varepsilon_{4,1}, \varepsilon_{4,2}, \ldots, \varepsilon_{m+2,1}$, and $\varepsilon_{m+2,2}$, using the consecutive integers from

$2m+2n+5$ to $4(m+n+1)$. Note that we have the following ordering of induced vertex sums over the vertices on outer cycle:

$$f^+(w_{1,1}) < f^+(w_{1,2}) < f^+(w_{2,1}) < f^+(w_{2,2}) < f^+(w_{1,3}) < f^+(w_{1,4}) < \cdots < f^+(w_{1,n+2}) \\ < f^+(w_{2,3}) < f^+(w_{2,4}) < \cdots < f^+(w_{2,n+2}) < f^+(w_{3,1}) < f^+(w_{3,2}) < f^+(w_{4,1}) \\ < f^+(w_{4,2}) < \cdots < f^+(w_{m+2,1}) < f^+(w_{m+2,2}).$$

Next we show that the vertex sums of vertices of T are all pairwise distinct in $P_{m+2} \times P_{n+2}$ for the above arranged edge labeling. First, we show by the following three cases that the vertex sums induced from the k-anti-magic labeling for T keep the same ordering in $P_{m+2} \times P_{n+2}$, although extra edges are appended to certain vertices.

Case 1: Let T_2 be the set of degree 2 vertices of T:

The vertices of $T_2 = \{w_{3,3}, w_{3,4}, w_{4,3}, w_{4,4}\}$ are of degree 2 in T, but of degree 4 in $P_{m+2} \times P_{n+2}$. Note that for these four vertices, the labels of the edges appended to them are in the same ordering with that of the induced vertex sums in T, hence the ordering keeps the same in $P_{m+2} \times P_{n+2}$.

Case 2: Let T_3 be the set of degree 3 vertices of T: Hence $T_3 = \{w_{3,5}, w_{3,6}, \ldots, w_{3,n+2}\} \cup \{(w_{4,5}, w_{4,6}, \ldots, w_{4,n+2}\} \cup \{w_{5,3}, w_{5,4}, w_{6,3}, w_{6,4}, \ldots, w_{m+2,3}, w_{m+2,4}\}$. The vertices of T_3 are of degree 3 in T, but of degree 4 in $P_{m+2} \times P_{n+2}$. The labels we assign to the edges $e_{1,5}, e_{1,6}, \ldots, e_{1,n+2}, e_{2,5}, e_{2,6}, \ldots, e_{2,n+2}, \varepsilon_{5,1}, \varepsilon_{5,2}, \varepsilon_{6,1}, \varepsilon_{6,2}, \ldots, \varepsilon_{m+2,1}, \varepsilon_{m+2,2}$ follow exactly the ordering of the vertex sum of the vertices $w_{3,5}, w_{3,6}, \ldots, w_{3,n+2}, w_{4,5}, w_{4,6}, \ldots, w_{4,n+2}, w_{5,3}, w_{5,4}, w_{6,3}, w_{6,4}, \ldots, w_{m+2,3}, w_{m+2,4}$, respectively. Hence the ordering keeps the same in $P_{m+2} \times P_{n+2}$.

Case 3: Let T_4 be the set of degree 4 vertices of T: For any vertex $w_{i,j}$ of T_4, the induced vertex sum is $f^+(w_{i,j}) = t^+(w_{i,j}) + 4k$.

Then, we will show the vertex sums of vertices of T are all pairwise distinct in $P_{m+2} \times P_{n+2}$. In fact, we have the following claims:

Claim 1. *The vertex sums of the vertices of T_2 are no greater than the vertex sums of the vertices of T_3 in $P_{m+2} \times P_{n+2}$.*

Claim 2. *The vertex sums of the vertices of T_3 are no greater than the vertex sums of the vertices of T_4 in $P_{m+2} \times P_{n+2}$.*

First we show Claim 1. In order to show this, we split into two cases. The first case is for $m \geqslant n = 2$, i.e. $P_{m+2} \times P_4$. The greatest vertex sum of T_2 is $f^+(w_{4,4}) = f(e_{2,4}) + f(\varepsilon_{4,2}) + t^+(w_{4,4}) + 2k$, and the smallest vertex sum of T_3 is $f^+(w_{5,3}) = f(\varepsilon_{5,1}) + t^+(w_{5,3}) + 3k$. Since T is anti-magic, $t^+(w_{4,4}) < t^+(w_{5,3})$. Hence $f^+(w_{4,4}) < f^+(w_{3,5})$ if and only if $f(e_{2,4}) + f(\varepsilon_{4,2}) \leqslant f(\varepsilon_{5,1}) + k$, and the latter inequality becomes $-1 \leqslant 2m$, which is true. The second case is for $m \geqslant n \geqslant 3$. The greatest vertex sum of T_2 is $f^+(w_{4,4}) = f(e_{2,4}) + f(\varepsilon_{4,2}) + t^+(w_{4,4}) + 2k$, and the smallest vertex sum of T_3 is $f^+(w_{3,5}) = f(e_{1,5}) + t^+(w_{3,5}) + 3k$. Also $t^+(w_{4,4})$ is smaller than $t^+(w_{3,5})$. Hence $f^+(w_{4,4}) < f^+(w_{5,3})$ if and only if $f(e_{2,4}) + f(\varepsilon_{4,2}) \leqslant f(e_{1,5}) + k$, and the latter inequality becomes $n + 3 \leqslant 2m$, which is true. So the Claim 1 is proved.

Now we show Claim 2. The greatest vertex sum of T_3 is $f^+(w_{m+2,4}) = f(\varepsilon_{m+2,2}) + t^+(w_{m+2,4}) + 3k$, and the smallest vertex sum of T_4 is $f^+(w_{5,5}) = t^+(w_{5,5}) + 4k$. Also $t^+(w_{m+2,4})$ is smaller than $t^+(w_{5,5})$, and $f(\varepsilon_{m+2,2})$ is smaller than k. Hence $f^+(w_{m+2,4})$ is smaller than $f^+(w_{5,5})$, and Claim 2 is established.

Finally, we will show the vertex sums of the vertices in $V(P_{m+2} \times P_{n+2}) - V(T)$ are smaller than the vertex sums of the vertices of T, by showing that $f^+(w_{m+2,2}) < f^+(w_{3,3})$. The inequality is actually $f(e_{m,2}) + f(e_{m+1,2}) + f(\varepsilon_{m+2,2}) < f(\varepsilon_{3,1}) + f(e_{1,3}) + t^+(w_{3,3}) + 2k$. We observe that $f(e_{m+1,2}) < f(e_{1,3})$, $f(e_{m,2}) < k$, and $f(\varepsilon_{m+2,2}) < k$, therefore the above inequality is true. Hence from all the above, we prove this theorem. □

Remark 10. In the process of writing up this article, Cheng [2] independently obtained the above result using a different approach. Here the mathematical induction is used, and with similar approaches the anti-magic-ness of higher dimensional lattice grid graphs $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_t}$ can be obtained, where t is an integer $\geqslant 3$.

4. Prism grids and toroidal grids are anti-magic

If G is a d-regular graph ($d \geqslant 1$), we call $G \times P_n$ ($n \geqslant 2$) a *generalized prism grid* graph. When G is a cycle, the graph $C_m \times P_n$ is called a prism grid graph. We show prism grid graphs are anti-magic by proving a more general fact that the generalized prism grid graphs are anti-magic, as in the following theorem.

Theorem 11. *The generalized prism grid graphs, i.e. the Cartesian product of paths P_n $(n \geqslant 2)$ and d-regular graphs $(d \geqslant 1)$, are anti-magic.*

Proof. We denote the graphs G to be a path P_n and H to be a d-regular graph with $|V(H)|=p, |E(H)|=q$. For the graph P_2, we have the same vertex sum on two vertices, i.e. $g^+(u_1)=g^+(u_2)$, where g is an edge labeling. For P_n, $n>2$, we label the edges by $g(e_i)=i$ for $1 \leqslant i \leqslant n-1$, such that the vertex sums have the order $g^+(u_1)<g^+(u_2)<\cdots<g^+(u_n)$. In general, with the edge labeling $h(\varepsilon_i)=i$, $1 \leqslant i \leqslant q$, we have the order $h^+(v_1) \leqslant h^+(v_2) \leqslant \cdots \leqslant h^+(v_p)$, and $h^+(v_p)-h^+(v_1) \leqslant d(q-d)<dq$. Then we label the edges of the slices H_k and G_l by:

$$f_H(\varepsilon_{k,j})=(k-1)(p+q)+h(\varepsilon_j) \quad \text{for } 1 \leqslant j \leqslant q, \quad 1 \leqslant k \leqslant n,$$
$$f_G(e_{i,l})=q+(p+q)(g(e_i)-1)+l \quad \text{for } 1 \leqslant i \leqslant n-1, \quad 1 \leqslant l \leqslant p.$$

Hence the vertex sum of the slices of H_k and G_l is

$$f_H^+(w_{k,j})=d(k-1)(p+q)+h^+(v_j),$$
$$f_G^+(w_{i,l})=\begin{cases} q+(p+q)(g^+(u_i)-1)+l, & i=1,2,\\ 2q+(p+q)(g^+(u_i)-2)+2l, & 3 \leqslant i \leqslant n.\end{cases}$$

We observe that the vertex sums f_H^+ and f_G^+ have the order $f_H^+(w_{i,1}) \leqslant f_H^+(w_{i,2}) \leqslant \cdots \leqslant f_H^+(w_{i,p})$ and $f_G^+(w_{i,1})<f_G^+(w_{i,2})<\cdots<f_G^+(w_{i,p})$ when i is fixed and ranged from 1 to n. So is $f^+(w_{i,1})<f^+(w_{i,2})<\cdots<f^+(w_{i,p})$, when i is fixed and ranged from 1 to n. Hence if we can show $f^+(w_{i,p})<f^+(w_{i+1,1})$ for i from 1 to $n-1$, it is anti-magic.

Now, we will discuss the anti-magic-ness of Cartesian products of paths and regular graphs in the following two cases.

Case 1: When $n=2$, i.e. $P_2 \times H$. The vertex sums are

$$f^+(w_{i,j})=f_G^+(w_{i,j})+f_H^+(w_{i,j})=f_G(e_{1,j})+f_H^+(w_{i,j})=q+j+d(i-1)(p+q)+h^+(v_j).$$

We observe that $f^+(w_{i,j})$ is strictly increasing when i fixes and j increases, hence if $f^+(w_{1,p})<f^+(w_{2,1})$ holds, this graph is anti-magic. But $f^+(w_{1,p})<f^+(w_{2,1}) \Leftrightarrow h^+(v_p)-h^+(v_1)<dq+dp-p+1$ and the latter is always true, so we are done.

Case 2: When $n \geqslant 3$. Note that

$$f^+(w_{1,p})<f^+(w_{2,1}) \quad \Leftrightarrow \quad h^+(v_p)-h^+(v_1)<dq+dp+(p+q)(g^+(u_2)-g^+(u_1))-p+1,$$

$$f^+(w_{2,p})<f^+(w_{3,1}) \quad \Leftrightarrow \quad h^+(v_p)-h^+(v_1)<dq+dp+q+(p+q)(g^+(u_3)-g^+(u_2)-1)-p+2,$$

and

$$f^+(w_{i,p})<f^+(w_{i+1,1}) \quad \Leftrightarrow \quad h^+(v_p)-h^+(v_1)<dq+dp+(p+q)(g^+(u_{i+1})-g^+(u_i)) -2p+2 \quad \text{for } 3 \leqslant i \leqslant n-1.$$

The above inequalities can be justified, hence the Cartesian products of paths and regular graphs are anti-magic. □

Corollary 12. *All prism grid graphs $C_m \times P_n$ are anti-magic.*

Corollary 13. *The hypercube graphs Q_n are anti-magic for $n \geqslant 2$.*

Proof. For $n \geqslant 3$, the hypercube Q_n is a graph isomorphic to $Q_{n-1} \times P_2$, and we have that Q_{n-1} is regular and Q_2 is isomorphic to C_4. □

If G is a d-regular graph $(d \geqslant 1)$, we call $G \times C_n$ $(n \geqslant 3)$ a *generalized toroidal grid* graph. When G is a cycle, the graph $C_m \times C_n$ is called a toroidal grid graph. In [8] has been proved that the generalized toroidal grid graphs are anti-magic.

Theorem 14. *The generalized toroidal grid graphs, i.e. the Cartesian products of cycles and regular graphs, are anti-magic.*

Therefore we have the corollaries that all toroidal grid graphs $C_m \times C_n$ are anti-magic for $m, n \geqslant 3$, and all higher dimensional toroidal grid graphs $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_t}$ are anti-magic for integer $t \geqslant 3$.

Note that we may get the above results by using the same labeling method as in proving Theorem 11, while the above results are obtained by a different approach in [8]. Also in the process of writing up this article, Cheng [2,3] independently generalizes the above results to the case of Cartesian product of regular graphs, and gives the anti-magic labeling of prism grids.

5. Cartesian products of graphs with regular graphs

With the similar ideas, we may obtain a more general situation as follows, which can be applied to find an anti-magic labeling of the Cartesian product of various types of graphs with regular graphs.

Theorem 15. *Assume G is a graph with $|V(G)| = p_1$ and $|E(G)| = q_1$. Without loss of generality by renaming the vertices, we may assume that $g^+(u_1) \leqslant g^+(u_2) \leqslant \cdots \leqslant g^+(u_{p_1})$ for a given edge labeling g. Let H be a d-regular graph with $|V(H)| = p_2$ and $|E(H)| = q_2$. If we have the inequalities*

$$p_2 \cdot [g^+(u_{i+1}) - g^+(u_i)] + p_1 \cdot q_2 \cdot [\deg(u_{i+1}) - \deg(u_i)] \geqslant (p_2 - 1) \cdot \deg(u_{i+1})$$

for $i = 1, 2, \ldots, p_1 - 1$, then the Cartesian product $G \times H$ is anti-magic.

Proof. Without loss of generality by renaming the vertices and edges, we may assume that $g^+(u_1) \leqslant g^+(u_2) \leqslant \cdots \leqslant g^+(u_{p_1})$ with the edge labeling $g(e_i) = i$ on G, $1 \leqslant i \leqslant q_1$, and with the edge labeling $h(\varepsilon_i) = i$ on H, $1 \leqslant i \leqslant q_2$, we have the ordering $h^+(v_1) \leqslant h^+(v_2) \leqslant \cdots \leqslant h^+(v_{p_2})$, and also $h^+(v_{p_2}) - h^+(v_1) < d \cdot q_2$. The notations here follow the conventions earlier mentioned. Therefore, we label the edges of the slices of H_k and G_l by

$$f_H(\varepsilon_{k,j}) = q_2(k-1) + h(\varepsilon_j) \quad \text{for } 1 \leqslant k \leqslant p_1, \ \ 1 \leqslant j \leqslant q_2,$$
$$f_G(e_{i,l}) = p_1 q_2 + p_2(g(e_i) - 1) + l \quad \text{for } 1 \leqslant i \leqslant q_1, \ \ 1 \leqslant l \leqslant p_2.$$

Hence the vertex sums of the slices of H_k and G_l are

$$f_H^+(w_{k,j}) = dq_2(k-1) + h^+(v_j) \quad \text{for } 1 \leqslant k \leqslant p_1, \ \ 1 \leqslant j \leqslant p_2,$$
$$f_G^+(w_{i,l}) = p_2 g^+(u_i) + \deg(u_i)(p_1 q_2 - p_2 + l) \quad \text{for } 1 \leqslant i \leqslant p_1, \ \ 1 \leqslant l \leqslant p_2.$$

The vertex sums f_H^+ and f_G^+ have the ordering $f_H^+(w_{i,1}) \leqslant f_H^+(w_{i,2}) \leqslant \cdots \leqslant f_H^+(w_{i,p_2})$ and $f_G^+(w_{i,1}) < f_G^+(w_{i,2}) < \cdots < f_G^+(w_{i,p_2})$, when i ranges from 1 to p_1. So is $f^+(w_{i,1}) < f^+(w_{i,2}) < \cdots < f^+(w_{i,p_2})$, when i ranges from 1 to p_1. Hence if we can show $f^+(w_{i,p_2}) < f^+(w_{i+1,1})$ for $1 \leqslant i \leqslant p_1 - 1$, then it is anti-magic. So we observe that $f^+(w_{i,p_2}) < f^+(w_{i+1,1}) \Leftrightarrow h^+(v_{p_2}) - h^+(v_1) < dq_2 + p_2[g^+(u_{i+1}) - g^+(u_i)] + p_1 q_2[\deg(u_{i+1}) - \deg(u_i)] - \deg(u_{i+1})(p_2 - 1)$, and the latter inequality is true whenever $p_2[g^+(u_{i+1}) - g^+(u_i)] + p_1 q_2[\deg(u_{i+1}) - \deg(u_i)] - \deg(u_{i+1})(p_2 - 1) \geqslant 0$. Hence we are done. □

Corollary 16. *Assume G is an anti-magic (p_1, q_1)-graph, and in addition that for the vertices $u_1, \ldots, u_{p_1}$ of G, we have consistent degree sequence and vertex sum ordering, i.e.* $\deg(u_1) \leqslant \deg(u_2) \leqslant \cdots \leqslant \deg(u_{p_1})$ *and $g^+(u_1) < g^+(u_2) < \cdots < g^+(u_{p_1})$ for an anti-magic edge labeling g. Suppose H is a d-regular (p_2, q_2)-graph. If we have the inequalities*

$$g^+(u_{i+1}) - g^+(u_i) \geqslant \deg(u_{i+1}),$$

for $i = 1, 2, \ldots, p_1 - 1$, then the Cartesian product $G \times H$ is anti-magic.

Proof. Since $\deg(u_{i+1}) - \deg(u_i)$ is non-negative and $g^+(u_{i+1}) - g^+(u_i) \geqslant \deg(u_{i+1})$ for $i = 1, 2, \ldots, p_1 - 1$, the inequalities in Theorem 15 are satisfied. □

With the above-mentioned conditions, we may get the anti-magic labeling of various types of graph products. Let the star graph $ST(n)$, $n \geqslant 1$, be the graph with $n + 1$ vertices, in which there is one center vertex incident to n degree 1 vertices. Let us give two examples below, using the star graphs $ST(n)$:

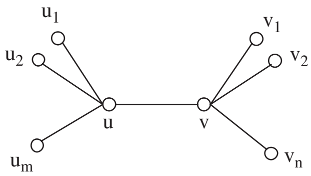

Fig. 5. Double star graph ST(m, n).

Corollary 17. *The Cartesian product of the star graphs* $ST(n)$, $n \geqslant 1$, *and regular graphs is anti-magic.*

Proof. Let $ST(n)$ be the star graph with the vertex set $\{u_1, \ldots, u_{n+1}\}$, and the vertex u_{n+1} is the one incident to the other n pendant vertices $u_1, \ldots, u_n$. If $n = 1$, then the Cartesian product is anti-magic by Theorem 11. If $n \geqslant 2$, notice that we have consistent degree sequence and vertex sum orderings as $\deg(u_1) = 1 \leqslant \cdots \leqslant \deg(u_n) = 1 \leqslant \deg(u_{n+1}) = n$ and $g^+(u_1) = 1 \leqslant \cdots \leqslant g^+(u_n) = n \leqslant g^+(u_{n+1}) = 1 + 2 + \cdots + n$ for the anti-magic edge labeling g with $g(u_i u_{n+1}) = i$ for $1 \leqslant i \leqslant n$. Hence the above inequalities in Corollary 16 can be easily checked. $\square$

The double star graph $ST(m, n)$ is a graph that is formed by two stars $ST(m)$ and $ST(n)$ via joining their centers by an edge (see Fig. 5). We give the anti-magic labeling of the Cartesian product of the double star graphs $ST(m, n)$ with regular graphs.

Corollary 18. *The Cartesian product of the double star graphs* $ST(m, n)$, *where* $(m, n \geqslant 1)$, *and regular graphs is anti-magic.*

Proof. Without loss of generality, we may assume that $m \leqslant n$ in the graph $ST(m, n)$. Let u and v be the center vertices of the graph $ST(m)$ and $ST(n)$, respectively, and let $u_1, u_2, \ldots, u_m$ be the neighbors of u, and $v_1, v_2, \ldots, v_n$ be the neighbors of v. Then we assign $1, 2, \ldots, m$ to the edges $uu_1, uu_2, \ldots, uu_m$, and assign $m + 1, m + 2, \ldots, m + n$ to the edges $vv_1, vv_2, \ldots, vv_n$, and assign $m + n + 1$ to the edge uv. Then we check whether it satisfies the sufficient conditions in Corollary 16. It is clear to see that over the vertices with degree 1 the inequalities are satisfied. So we only need to check two inequalities $f^+(u) - f^+(v_n) \geqslant \deg(u)$ and $f^+(v) - f^+(u) \geqslant \deg(v)$. For the first one, $f^+(u) - f^+(v_n) = m(m + 1)/2 + 1$, it must be greater or equal to $\deg(u) = m + 1$. For the second inequality, $f^+(v) - f^+(u) = (2mn + (n^2 + n - m^2 - m))/2 \geqslant mn$ is greater than $n + 1$ if $n \geqslant m \geqslant 2$, and $f^+(v) - f^+(u) = (2n + (n^2 + n - 2))/2$ is greater than $n + 1$ if $n > m = 1$. Otherwise, if $m = n = 1$, then $ST(1, 1)$ is isomorphic to P_4, and according to Theorem 11, the Cartesian product of $ST(1, 1)$ and regular graphs is anti-magic. So we are done. $\square$

Using Theorem 15 and Corollary 16, more examples of anti-magic cartesian products can be obtained.

6. Lexicographic product of graphs

Given two graph $G = (V_1, E_1)$ and $H = (V_2, E_2)$. The *lexicographic product* of G and H is the graph $G[H]$ with vertex set $V_1 \times V_2$ and for $(u_1, v_1), (u_2, v_2) \in V_1 \times V_2$, and (u_1, v_1) is adjacent with (u_2, v_2) whenever (1) $u_1 = u_2$ and v_1 is adjacent to v_2 in H or (2) u_1 is adjacent to u_2 in G. $G[H]$ is also called the *composition* of G and H.

It is well-known that $G[H]$ and $H[G]$ are usually not the same in general. Let $G(V_1, E_1)$ be a (p_1, q_1)-graph, where $|V_1| = p_1$, $|E_1| = q_1$, and $H = (V_2, E_2)$ be a (p_2, q_2)-graph, where $|V_2| = p_2$, and $|E_2| = q_2$. Then we notice that the lexicographic product $G[H]$ can be decomposed into p_1 isomorphic copies of H, namely the subgraphs H_i such that $E(H_i) = \{(u_i, v_j)(u_i, v_k) : v_j v_k \in E_2\}$ for $1 \leqslant i \leqslant p_1$, together with q_1 isomorphic copies of complete bipartite graphs K_{p_2, p_2}, namely the subgraphs K_j such that $E(K_j) = \{(u_s, v_m)(u_t, v_n) : u_s u_t = e_j\}$ for $e_j \in E_1$, where $1 \leqslant j \leqslant q_1$.

On the other hand, the anti-magic labeling of the lexicographic product $G[H]$ is related to the theory of magic labeling. The notion of *super-magic graphs* was introduced by Stewart [7] in 1966. A (p, q)-graph G is called super-magic if it admits an edge labeling by pairwise distinct consecutive positive integers $1, 2, \ldots, q$ such that the sum of the labels of the edges incident with a vertex is the same among all vertices. The classic concept of $n \times n$ magic square in number theory corresponds to the super-magic labeling of $K_{n,n}$ for $n > 2$. In the following, we make use of the super-magic labeling of $K_{n,n}$ to create an anti-magic labeling of the lexicographic product of graphs.

Theorem 19. *If $G = (V_1, E_1)$ is a (p_1, q_1)-graph and $H = (V_2, E_2)$ is an anti-magic d-regular (p_2, q_2)-graph with $d > 1$, then $G[H]$ is anti-magic.*

Proof. Let H_i, $1 \leqslant i \leqslant p_1$, be the isomorphic copies of the subgraph H of $G[H]$, and K_j, $1 \leqslant j \leqslant q_1$, be the isomorphic copies of the complete bipartite subgraph K_{p_2,p_2} in the decomposition of $G[H]$ as stated above. Let f be the edge labeling on $G[H]$ with induced vertex sum $f^+ = f_K^+ + f_H^+$, where f_K^+ is the vertex sum restricted to those copies of complete bipartite subgraphs K_j, and f_H^+ is the vertex sum restricted to those copies of H_i. Since complete bipartite graphs are super-magic [7], we may label the edges of K_j by using consecutive positive integers from 1 to $q_1 \cdot (p_2)^2$ so that we have the same vertex sum contribution to the vertices (u, v_i), where $u \in V_1$, $v_i \in V_2$, and $1 \leqslant i \leqslant p_2$. Without loss of generality, we may assume, via renaming the vertices, that $f_K^+(u_1, v) \leqslant f_K^+(u_2, v) \leqslant \cdots \leqslant f_K^+(u_{p_1}, v)$.

We now complete the anti-magic labeling for $G[H]$ by assigning the labels $p_2^2 \cdot q_1 + 1, p_2^2 \cdot q_1 + 2, \ldots, p_2^2 \cdot q_1 + p_1 \cdot q_2$ to those edges of $H_i's$, where $1 \leqslant i \leqslant p_1$. Since H is anti-magic, we have a labeling $h : E_2 \to \{1, 2, \ldots, q_2\}$ such that $h^+(v_1) < h^+(v_2) < \cdots < h^+(v_{p_2})$ where h^+ is the induced vertex sum of h. Since H is also regular, by shifting the edge labeling we may assign $p_2^2 \cdot q_1 + (k-1) \cdot q_2 + h(v_i v_j)$ to the edges $(u_k, v_i)(u_k, v_j)$, such that $f_H^+(u_i, v_1) < f_H^+(u_i, v_2) < \cdots < f_H^+(u_i, v_{p_2})$, where $1 \leqslant i \leqslant p_1$, and also $f_H^+(u_i, v_{p_2}) < f_H^+(u_{i+1}, v_1)$, where $1 \leqslant i \leqslant p_1 - 1$. Therefore we have

$$\begin{aligned} f_K^+(u_1, v_1) &= f_K^+(u_1, v_2) = \cdots = f_K^+(u_1, v_{p_2}) \leqslant f_K^+(u_2, v_1) = f_K^+(u_2, v_2) \\ &= \cdots = f_K^+(u_2, v_{p_2}) \leqslant \cdots \leqslant f_K^+(u_{p_1}, v_1) = f_K^+(u_{p_1}, v_2) = \cdots = f_K^+(u_{p_1}, v_{p_2}) \end{aligned}$$

and

$$\begin{aligned} f_H^+(u_1, v_1) &< f_H^+(u_1, v_2) < \cdots < f_H^+(u_1, v_{p_2}) < f_H^+(u_2, v_1) < f_H^+(u_2, v_2) \\ &< \cdots < f_H^+(u_2, v_{p_2}) < \cdots < f_H^+(u_{p_1}, v_1) < f_H^+(u_{p_1}, v_2) < \cdots < f_H^+(u_{p_1}, v_{p_2}). \end{aligned}$$

Since the vertex sum f^+ on vertices of $G[H]$ is $f_K^+ + f_H^+$, it is easy to see from above inequalities that all the vertex sums of $G[H]$ are pairwise distinct, hence it is anti-magic. □

From the theorem above, we clearly have the following:

Corollary 20. *The composition graphs $P_m[C_n]$ and $C_m[C_n]$ are anti-magic.*

7. Future studies

While we have obtained the above results on graph products, it will be very interesting if one can verify the Hartsfield–Ringel conjecture by showing the anti-magic-ness of sparse graphs, contrast to the result of Alon et al. [1]. In particular, inspired from our results here, the anti-magic-ness of regular graphs is still unknown and is nice to explore. Many special classes of regular graphs have been investigated [6,8,2,3], and it is evident that we need the last piece for the jigsaw puzzle.

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